Violation of Bell's Inequality with Transverse Spatial Variables Using Fractional Fourier Transforms

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We report a novel Bell's inequality experiment using optical fractional Fourier transforms of transverse spatial degrees of freedom of photon pairs. Simple optical lens systems were used to implement variable-order fractional Fourier transforms of an input plane, while the detection plane was divided into two regions, resulting in a variable dichotomic detection system. We obtained a violation of the Clauser-Horne-Shimony-Holt inequality of more than 14 standard deviations.

Explanation of the strange counter-intuitive character of correlated quantum systems through hidden variable theories was first suggested by Einstein, Podolsky and Rosen in their seminal "EPR" paper [1]. In 1965, John Bell developed a method to test the predictions of quantum mechanics against classical hidden variable theories [2], and consequently, violation of Bell's inequality by a pair of correlated systems indicates that they exhibit a stronger-than-classical correlation, and are thus entangled. Shortly thereafter, Bell's inequalities were recast in such a way as to allow for direct experimental verification. In particular, for any classical hidden variable theory, the Clauser-Horne-Shimony-Holt (CHSH) inequality [3] predicts $-2 \leq \mathcal{S} \leq 2$, where $\mathcal{S} = E(\alpha_1, \alpha_2) + E(\alpha_1, \alpha_4) + E(\alpha_3, \alpha_2) - E(\alpha_3, \alpha_4)$,

$$E(\alpha, \beta) = \frac{C(\alpha, \beta) + C(\bar{\alpha}, \bar{\beta}) - C(\bar{\alpha}, \beta) - C(\alpha, \bar{\beta})}{C(\alpha, \bar{\beta}) + C(\bar{\alpha}, \bar{\beta}) + C(\bar{\alpha}, \beta) + C(\alpha, \bar{\beta})} \quad (1)$$

is the correlation function, and $C(\alpha, \beta)$ are the number of counts for analyzer settings α and β. The CHSH inequality in this form is applicable to dichotomic observables, so that a measurement in the α basis gives either result α or $\bar{\alpha}$ with respective probabilities $P(\alpha)$ and $P(\bar{\alpha})$, and $P(\alpha) + P(\bar{\alpha}) = 1$.

For certain situations, quantum mechanics predicts $2 < |\mathcal{S}| \leq 2\sqrt{2}$, and consequently, there has been much experimental effort to put quantum theory to the test, beginning with the early work of Freedman and Clauser [4] and later Aspect, Grangier and Roger [5]. These early experiments relied on polarization-entangled photons obtained from an atomic cascade. The development of spontaneous parametric down-conversion (SPDC) as a robust source of entangled photons has led to a number of experimental tests of Bell's inequalities in a number of discrete degrees of freedom (DOF), including polarization [6, 7, 8, 9, 10], dual-rail momentum modes [11], time-bin [12, 13] and orbital angular momentum [14]. These DOF have also been well-exploited in quantum information experiments [15], showing that entanglement is indeed a resource useful for information processing.

At the same time, there has been much research in quantum imaging [16, 17, 18, 19] using continuous-variable transverse spatial degrees of freedom of photon pairs obtained from SPDC. Recently, it was shown that these photon pairs are entangled in transverse spatial DOF through violation of a separability criterion involving the near-field (position) and far-field (momentum) distributions [20, 21]. Whereas these experiments show that photon pairs obtained from SPDC are entangled, there remains the question: *is it possible to violate Bell's inequalities with transverse spatial DOF of photon pairs?* One might believe that the answer to this question is negative, since it is known that the quantum state describing the position and momentum of SPDC photon pairs is similar to the EPR state [20, 21], which can be described by a positive-valued Wigner function. It has been argued that this prohibits violation of Bell's inequality in these particular degrees of freedom [22]. Though it has been shown that states described by positive Wigner functions can indeed be used to violate Bell's inequality [23, 24], these demonstrations use different degrees of freedom of optical fields.

In this letter, we answer the above question in the affirmative by reporting a novel Bell's inequality experiment using transverse spatial properties of photon pairs produced by SPDC. Using variable analyzers composed of optical fractional Fourier transform (FRFT) systems [25, 26, 27, 28] and a dichotomized detection region, we obtained an experimental violation of the CHSH inequality of more than 14 standard deviations: $\mathcal{S} = 2.44 \pm 0.03$.

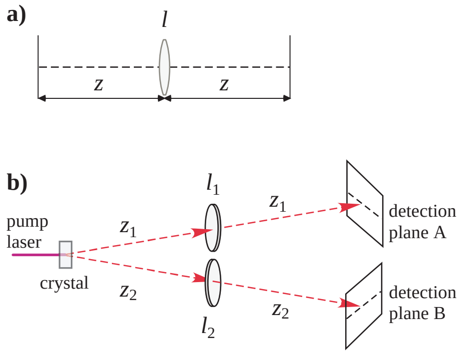

FIG. 1: a) An optical FRFT system. l is the focal length of the lens, and z is defined in the text. b) Sketch of the experiment.

The FRFT of an optical field can be observed through ordinary Fresnel diffraction, in the same way that the usual Fourier transform is observed in Fraunhoffer diffraction [25]. Mathematically, the FRFT of a function $f(x)$ is denoted $\mathcal{F}_\phi[f(x)]$, where ϕ is the degree of the transform [26, 27]. It has been shown that one can implement an optical FRFT of order $\phi \in [0, \pi]$ in a controlled manner using a simple lens system as shown in FIG. 1 a), where the distance $z = 2l\sin^2(\phi/2)$, and l is the focal length of the lens [28]. Using the additivity property of the FRFT: $\mathcal{F}_\phi\mathcal{F}_\varphi[f(x)] = \mathcal{F}_{\phi+\varphi}[f(x)]$, orders larger than π can be implemented using a series of lenses [26].

An illustration of the basic idea of our experiment is shown in FIG. 1 b). Signal and idler photons produced by SPDC in a non-linear crystal pass through separate optical FRFT systems and are detected in detection planes A and B, respectively. In order to use Bell-type inequalities, it is necessary to dichotomize the continuous spatial variables by dividing each of the detection planes into an upper and lower half. We will identify ϕ^+ and ϕ^- as measurements using a ϕ-order optical FRFT and detection in the upper and lower detection region, in analogy to α and $\bar{\alpha}$ in Eq. (1). In this way we account for all possible detection events: $P(\phi^+) + P(\phi^-) = 1$.

We will outline the theory for this experiment, the complete details can be found in Ref. [29]. The probability to detect one photon at transverse position $\boldsymbol{\rho_s}$ and one at $\boldsymbol{\rho_i}$ in the detection regions is $P_{\phi_s,\phi_i}(\boldsymbol{\rho}_s, \boldsymbol{\rho}_s) = |\Psi(\phi_s, \phi_i)|^2$, where $\Psi_{\phi_s,\phi_i}(\boldsymbol{\rho}_s, \boldsymbol{\rho}_s) = \langle \mathrm{vac}| \mathsf{E}(\boldsymbol{\rho}_s)\mathsf{E}(\boldsymbol{\rho}_i) |\psi\rangle$ is the detection amplitude and $\mathsf{E}_\phi(\boldsymbol{\rho})$ is the field operator for an FRFT system of order ϕ. In the thin crystal and monochromatic approximations, the post-selected two-photon state $|\psi\rangle$ at the crystal face is [19]

$$|\psi\rangle = \iint d\mathbf{q}_s d\mathbf{q}_i v(\mathbf{q}_s + \mathbf{q}_i)\gamma(\mathbf{q}_s - \mathbf{q}_i) |\mathbf{q}_s\rangle |\mathbf{q}_i\rangle . \quad (2)$$

Here $v(\mathbf{q})$ is the angular spectrum of the pump beam at the crystal face, and $\gamma(\mathbf{q})$ is the phase matching function of the non-linear crystal. Vectors $\mathbf{q}_s$ and $\mathbf{q}_i$ are the transverse components of the wave vectors of the down-converted photons. For simplicity, we will assume that the down-converted photons are degenerate, that is $k_s = k_i = K/2$, where K is the magnitude of the pump beam wave vector, in which case $\gamma(\mathbf{q}) = \sqrt{2L/\pi^2 K}\mathrm{sinc}(L|\mathbf{q}_s - \mathbf{q}_i|^2/4K)$. Using lens systems as shown in 1 a), the operator $\mathsf{E}_\phi(\boldsymbol{\rho})$ is

$$\mathsf{E}_\phi(\boldsymbol{\rho}) = \exp\left(-i\frac{k\tan\phi}{2f}\rho^2\right) \times \int d\mathbf{q} \exp\left(-i\frac{f\tan\phi}{2k}q^2\right) \exp\left(i\frac{\boldsymbol{\rho}\cdot\mathbf{q}}{\cos\phi}\right) \mathsf{a}(\mathbf{q}), \quad (3)$$

where $f = l\sin\phi$ is the scaled focal length of the lens with focal length l [28]. In order to choose the FRFT orders used to test the CHSH inequality, it is illustrative to make a direct analogy with polarization-based Bell experiments, in which maximum violation is achieved when each user uses a set of conjugate bases. It is well known that imaging ($\phi = \pi$) and Fourier ($\phi = \pi/2$) are conjugate lens systems [30, 31]. We will thus consider $\phi_i = \{\pi/2, \pi\}$ and consider the detection probabilities $P_{\phi_s,\frac{\pi}{2}}(\boldsymbol{\rho}_s, \boldsymbol{\rho}_s)$ and $P_{\phi_s,\pi}(\boldsymbol{\rho}_s, \boldsymbol{\rho}_s)$.

Let us assume that the angular spectrum $v(\mathbf{q})$ of the pump beam is much narrower than the phase matching function $\gamma(\mathbf{q})$. Then, the detection probability $P_{\phi_s,\frac{\pi}{2}}(\boldsymbol{\rho}_s, \boldsymbol{\rho}_s)$ for a Gaussian pump beam $v(\mathbf{q}) = \exp(-q^2/2\sigma^2)$ is [29]

$$P_{\phi_s,\frac{\pi}{2}}(\boldsymbol{\rho}_s, \boldsymbol{\rho}_i) = \exp\left[-\sigma^2 \frac{(\boldsymbol{\rho}_s + \sin\phi_s \boldsymbol{\rho_i})^2}{\cos^2\phi_s + 4\sigma^4 f^2 \sin^2\phi_s/K^2}\right]. \quad (4)$$

We will approximate the phase matching function $\gamma(\boldsymbol{\eta})$ by a Gaussian function of the form $\gamma(\boldsymbol{\eta}) = \exp(-\delta^2\eta^2/2)$. Then the detection probability $P_{\phi_s,\pi}(\boldsymbol{\rho}_s, \boldsymbol{\rho}_s)$ is [29]

$$P_{\phi_s,\pi}(\boldsymbol{\rho}_s, \boldsymbol{\rho}_i) = \exp\left[-\frac{\Delta^2(\boldsymbol{\rho}_s + \cos\phi_s \boldsymbol{\rho_i})^2}{\delta^2(\sin^2\phi_s + 4\Delta^4\cos^2\phi_s)}\right], \quad (5)$$

where $\Delta = \delta^2 K/f$.

The coincidence count rates $C(\phi_s, \frac{\pi}{2})$ and $C(\phi_s, \pi)$ for detectors equipped with half plane apertures are given by integrating Eqs. (4) and (5) over the areas defined by the apertures. Numerical integration shows a violation of the CHSH inequality for typical experimental parameters. We have chosen the four sets of FRFT measurements $\{\phi_1^+, \phi_1^-\} = \{\pi^+, \pi^-\}$, $\{\phi_2^+, \phi_2^-\} = \{\frac{5\pi}{4}^+, \frac{5\pi}{4}^-\}$, $\{\phi_3^+, \phi_3^-\} = \{\frac{\pi}{2}^+, \frac{\pi}{2}^-\}$, $\{\phi_4^+, \phi_4^-\} = \{\frac{3\pi}{4}^+, \frac{3\pi}{4}^-\}$, where, for example, a ϕ_1^+ measurement is a π-order FRFT and detection in the upper spatial region, and ϕ_1^- as a π FRFT and detection in the lower spatial region. Considering that the pump beam profile is a Gaussian with width $w = 4$mm, the width of the angular spectrum is $\sigma = 1/w = 0.25\text{mm}^{-1}$. The width of the function $\gamma(\mathbf{q})$, measured for a 5mm long $LiIO_3$ crystal, using 10nm FWHM interference filters is about 200mm^{-1} [32]. Using $K = 14 \times 10^6\,\text{mm}$, $l = 200\,\text{mm}$, and considering a total detection region of 12 mm, through numerical integration of Eqs. (4) and Eqs. (5) we obtain $E(\frac{5\pi}{4}, \pi) = 0.73$, $E(\frac{3\pi}{4}, \pi) = 0.73$, $E(\frac{5\pi}{4}, \frac{\pi}{2}) = 0.61$, and $E(\frac{3\pi}{4}, \frac{\pi}{2}) = -0.61$, giving $\mathcal{S} = 2.68$, a predicted violation of the Bell-CHSH inequality.

Our experimental setup is shown in FIG. 2. A 200mW He-Cd laser ($\lambda = 441.6$ nm) first passed through a $\times 3$ beam expander and was then used to pump a 5 mm long $LiIO_3$ crystal cut for type I phase matching. The expander was adjusted so that the beam waist was positioned 110 cm from the crystal face. The detectors were Perkin Elmer photon counting modules equipped with 10 nm FWHM interference filters centered around 884 nm. Initially, with all lenses removed, detectors A and B were aligned and optimized around two-photon coincidence detections. The optical FRFT systems were mounted on translation stages so that they could be toggled in and out of the paths of the down-converted photons. The ϕ_1 and ϕ_2 FRFTs were composed of two FRFTs, $\phi_1 = \phi_{11} + \phi_{12}$ and $\phi_2 = \phi_{21} + \phi_{22}$, as shown in FIG 2. The focal lengths and z-distances used are shown in table I. In order to keep the detectors stationary and toggle only the lenses, we chose focal lengths and FRFT orders that no longer correspond to perfect additivity (f constant) nor maximal Bell's inequality violation. Although one can easily implement the ideal case by adjusting the longitudinal positions of the lenses and detection planes, we opted to accept these imperfections in exchange for a very simple experimental setup, in addition to fixed reference positions for coincidence detections.

Each detector was completely open, resulting in a 12 mm diameter circular detection region. The upper and lower portions of the detection regions were selected using 12 mm $\times$ 6 mm rectangular slits. The slits were placed immediately before the detectors and were mounted on swiveling mirror mounts so that they could easily be removed from the setup. Micrometer translation stages were used to fine-tune the position of the slits.

We performed an initial calibration step for each combination of FRFT's ϕ_j-ϕ_k ($j = 1, 3$, $k = 2, 4$), in which we measured the coincidence counts corresponding to the upper and lower position, $C(\phi_j^+, \phi_k^0)$ and $C(\phi_j^-, \phi_k^0)$, with one blade swiveled out of the setup (which we will call a "0" measurement), The blade was carefully aligned so that $C(\phi_j^+, \phi_k^0)$ and $C(\phi_j^-, \phi_k^0)$ were equal to half the total coincidence counts $C(\phi_j^0, \phi_k^0)$, obtained with both blades removed. Care was taken so that the single photon counts corresponding to each slit position were also half of the total counts when the slit was removed. We repeated this procedure for all FRFTs in both paths, obtaining all possible combinations of $C(\phi_j^+, \phi_k^0)$, $C(\phi_j^-, \phi_k^0)$, $C(\phi_j^0, \phi_k^+)$ and $C(\phi_j^0, \phi_k^-)$.

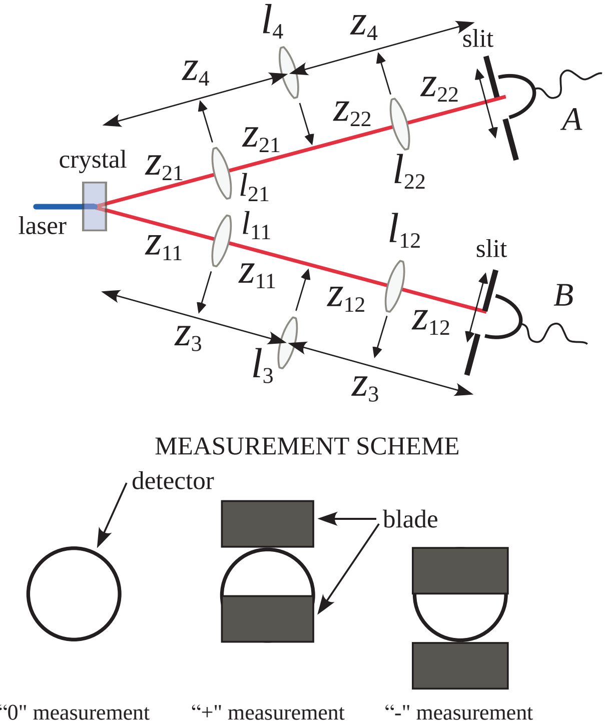

FIG. 2: Experimental setup and measurement scheme.

TABLE I: Focal lengths f and distances z for optical FRFTs of order ϕ.

ϕ	l	z
$\phi_{11} = \frac{\pi}{2}$	l_{11}=25.0cm	z_{11} =25.0cm
$\phi_{12} = \frac{29\pi}{50}$	l_{12} =20.0cm	z_{12} =25.0cm
$\phi_{21} = \frac{\pi}{2}$	l_{21}=25.0cm	z_{21} =25.0cm
$\phi_{22} = \frac{3\pi}{4}$	l_{22} =15.0cm	z_{22} =25.6cm
$\phi_3 = \frac{\pi}{2}$	l_3 = 50.0cm	z_3 = 50.0cm
$\phi_4 = \frac{37\pi}{50}$	l_4 = 30.0cm	z_4 = 50.6cm

The experimental results are shown in FIG. 3. Error bars are not shown, but, following the usual procedure for Poissonian statistics [33], correspond to the square root of the number of coincidence counts. One can see that the coincidence counts for all $C(\phi_j^\pm, \phi_k^0)$ and $C(\phi_j^0, \phi_k^\pm)$ calibration measurements are approximately half of the total $C(\phi_j^0, \phi_k^0)$ counts. We measured coincidences for all four possible combinations of detection regions for each possible FRFT configuration, thus obtaining $C(\phi_j^+, \phi_k^+)$, $C(\phi_j^-, \phi_k^-)$, $C(\phi_j^+, \phi_k^-)$ and $C(\phi_j^-, \phi_k^+)$. Using the values shown, we calculated the correlation functions $E(\phi_j, \phi_k)$ given by Eq. (1), and then the value of $\mathcal{S}$. We obtained $E(\phi_1, \phi_2) = 0.44 \pm 0.02$, $E(\phi_1, \phi_4) = 0.62 \pm 0.01$, $E(\phi_3, \phi_2) = 0.71 \pm 0.01$ and $E(\phi_3, \phi_4) = -0.67 \pm 0.01$, giving $\mathcal{S} = 2.44 \pm 0.03$, a violation of more than 14 standard deviations. We note that we obtain a considerable

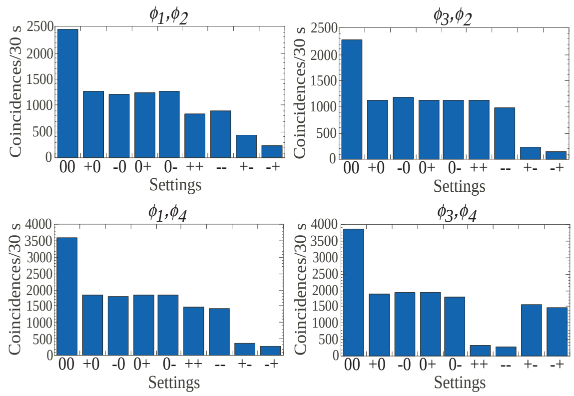

FIG. 3: Coincidence counts for all FRFT configurations.

violation even though we have relaxed the conditions on the analyzer settings, which indicates that this type of Bell inequality test is fairly robust.

Using optical fractional Fourier transforms, we have shown that transverse spatial degrees of freedom of photon pairs contain sufficient entanglement to demonstrate quantum nonlocality by violation of the CHSH inequality. Our results show that transverse spatial properties of photons can be put to use in quantum information schemes, such as quantum cryptography based on Bell's theorem [30, 34]. Our experiment demonstrates that it is possible to dichotomize continuous variable degrees of freedom and violate a Bell's inequality, which suggests that the same procedure might be implemented in other continuous variable systems. The setup is easily extendable to higher dimensional systems, allowing for the test of different types of Bell inequalities, more resistant to noise [35], as well as quantum key distribution in higher dimensional alphabets [31].

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